

Analysis of generalized synchronization in directionally coupled chaotic phase-coherent oscillators by local minimal fluctuations

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The method of local-minimum fluctuation is proposed to analyze generalized synchronization in directionally coupled chaotic phase-coherent oscillators. It is shown that the emergence of generalized synchronization is manifested by the qualitative changes in the statistic of local minimum fluctuations of the receiver oscillator.

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Synchronization phenomena in coupled or driven chaotic systems have been extensively studied in recent years [1–9]. Different notions of synchronization, e.g., identical synchronization, generalized synchronization (GS), phase synchronization (PS), and lag synchronization (LS) have been introduced in order to explore the qualitative intrinsic dynamical changes caused by the onset of chaos synchronization in experimental studies. Among these various types of notions, identical synchronization has been exhaustively investigated in relation to the stability of the synchronized manifold. Studies of GS and PS are also motivated by the need for a better understanding of the complex behaviors found in biological systems [7–9]. These explorations are closely related to the control of chaos and pattern formations in spatiotemporal systems, where they have revealed rich complex order embedded in chaotic motions.

Generalized synchronization extends the idea of identical synchronization to cases of directionally coupled systems (the so-called drive-response systems) with nonidentical individual dynamics. Recently, GS has been extended to discussions of bidirectionally arrays of coupled chaotic systems [10]. As it is defined in Ref. [1], the onset of GS in drive-response chaotic systems corresponds to the formation of a continuous mapping that transforms the trajectory on the attractor of the drive system into that on the attractor of the response system. Several methods have been presented to detect and analyze GS.

(i) The *auxiliary-system method* [1] was proposed as a practical method for identifying such a continuous mapping by being able to persistently point out the current state of the response system without direct computation of the map. This method requires the introduction of a third (auxiliary) system that is an exact replica of the response system. GS yields a chaotic attractor in the invariant manifold $X_r = X_a$, where X_r and X_a are variables corresponding to the response and auxiliary systems, respectively.

(ii) The method of *Poincaré cross sections* [2] helps to detect the onset of topological equivalence between the attractors of the synchronized drive and response systems.

(iii) The Lyapunov exponents of the response system, i.e., the *conditional Lyapunov exponents* are also useful diagnostic tools. Generalized synchronization is achieved if the largest conditional Lyapunov exponent is negative [except for some special cases, e.g., the possibility of period-doubled synchronization].

(iv) Due to the existence of a map relationship between the driver and receiver, many properties of the trajectories of the synchronized chaotic attractor in the embedding space of the receiver (R_E) and of the related trajectories in the embedding space of the driver (D_E) should be similar. The idea of *mutual false nearest neighbors* [1], which tests whether neighborliness in D_E translates in a practical numerical sense to neighborliness in R_E , has been used as a tool to study GS.

(v) An approach based on symbolic analysis has been presented in Ref. [4]. GS appears when an appropriately defined *conditional entropy* has a sharp minimum.

The exploration of the relation among various types of synchronizations is a subtle issue. Apart from identical synchronization that requires a complete match of the manifolds of subsystems, a clear distinction of other forms of synchronization is often difficult and remains an open subject. GS is still a concept that lacks clear definition, and the manner it relates to other synchronization types is still not quite clear. For phase-coherent chaotic oscillators, Parlitz *et al.* [3] claimed that in general the presence of GS always implies PS if one can define a *suitable* phase variable. It is not clear what is the criterion for the *suitable* phase variable. One usually adopts oscillator systems (e.g., Rössler) with mismatched parameters in studies of GS. It has been shown that for small parameter misfits, GS is identified as lag synchronization (the relation between the driver and the receiver is a time shift on the same attractor), and GS implies PS, i.e., PS is usually obtained before GS as coupling parameter is increased; For moderate parameter misfit, GS does neither correspond to LS and nor always imply PS, at least not PS as it is conventionally defined [5]. These distinctions are issues we deal with further here.

The detection of GS in practice is an important problem. The above methods of detecting GS are all based on a comparison between the drive and the response. However, under many circumstances one does not know (or does not need to know) the information from the drive subsystem (or a *black box*). Can one detect the synchronization information by only resorting to the receiver? To our knowledge, detection of GS based on the time series of the receiver has not been addressed previously. Such a possibility should be interesting and useful. It may help us understand what happens in the receiver when there is GS for situations when one can only obtain the time series of the receiver. In this paper, we pro-

pose the local-minimum fluctuation (LMF) statistics to assist in detecting GS. In practice precalibration of the system is first required to make the test most successful. This approach is practical and valid for different model systems we have studied. Moreover, as shown below, we can use this approach to explain why there are difficulties in showing whether GS does or does not imply PS, a topic that has been of interest in the literature of late.

A first sketch of the relationship between PS and GS has been given in Ref. [5] for the case of drive-response Rössler and Lorenz systems. This study only examined the situation $\omega_d < \omega_r$, where ω_d is the frequency of the drive oscillator and ω_r is the frequency of the receiver. Below we extend this work by considering both cases $\omega_d > \omega_r$ and $\omega_d < \omega_r$. There appears to be a difference between these two cases.

We focus on the situation of phase-coherent oscillators with moderate parameter misfit where, according to Ref. [5], the presence of GS does not always imply the presence of conventionally defined PS [5]. This in fact conflicts with a speculation of Parlitz [3]. Our method of examining GS in this regime is based on LMF's in the time series of the receiver. We found that when the mismatch of the drive-response system becomes too large, GS is accompanied with the emergence (or disappearance) of LMF in the time series of the receiver. The LMF arises in the response system as part of the process of adjusting and synchronizing (GS) to the drive system's significantly different frequency. The LMF tends to appear in a manner that permits GS but gives little clear indication of the phase locking associated with PS. Note that in the situation where the misfit between parameters of the receiver and driver is relatively small and GS implies PS (and LS), as shown in Ref. [5], we find no emergence or disappearance of the LMF. In this regime the phase locking associated with PS is not difficult to achieve.

We first study the drive-response Rössler oscillators with parameter misfits ($\omega_{d,r} = 1.0 \pm 0.03$):

$$\begin{aligned}
 \dot{x}_d &= -\omega_d y_d - z_d, \\
 \dot{y}_d &= \omega_d x_d + 0.15 y_d, \\
 \dot{z}_d &= 0.2 + z_d(x_d - 10.0), \\
 \dot{x}_r &= -\omega_r y_r - z_r + g(x_d - x_r), \\
 \dot{y}_r &= \omega_r x_r + 0.15 y_r, \\
 \dot{z}_r &= 0.2 + z_r(x_r - 10.0).
 \end{aligned} \tag{1}$$

The fourth-order Runge-Kutta method with fixed time steps 0.01 is used for all integrations. We analyzed the statistics on the amplitude of the LMF in the time series ($\ln z$) of the receiver. The reason we chose z ($\ln z$) of the receiver (Rössler and Foodweb model below) is that (i) more information can be detected easily from time series of $\ln z$ [11] and (ii) z is similar to some biological time series (Lynx Canada, phytoplankton in lakes and number of cases of measles in New York City [8,9]). One needs to transfer the time series ($\ln z$)

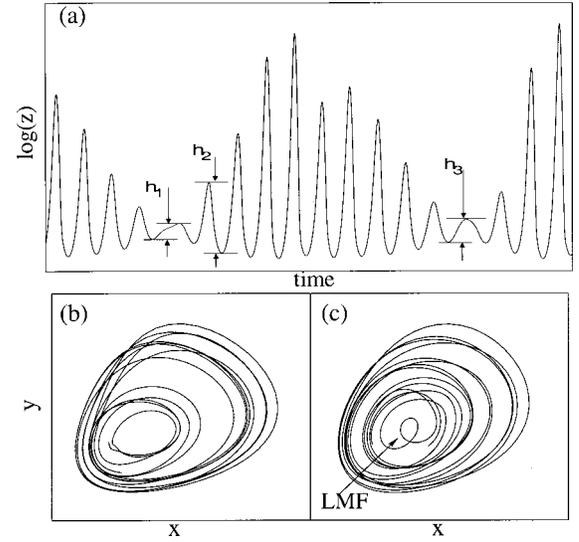


FIG. 1. The schematic description of local-minimum fluctuation. h_1 and h_3 are local-minimal fluctuations, but h_2 is not (a). The projection of the foodweb attractor on plane x - y before GS (b) and around GS critical point (c). In (c) a new LMF is pointed out.

to the amplitude fluctuation times series as the difference between each adjacent local extremal values [a local minimum (or maximum) and a local maximum (or minimum)] in the time series ($\ln z$) as shown in Fig. 1(a). Then one obtains a time series of the local minimal values of amplitude fluctuation time series and their statistics may be analyzed. In Fig. 1(a), we give a section of time series ($\ln z$) of the receiver where we identify three fluctuation values h_1 , h_2 , and h_3 . It is important to note that h_1 and h_3 are true local-minimal values. In contrast h_2 is not a local-minimum value—its neighboring peak heights indicate that this is not a local minimum. The statistics are based on values such as h_1 and h_3 . The projections of the receiver's orbit on x - y plane before GS and around GS critical point are given in Fig. 1(b) and 1(c), respectively. The difference is clear. There is one LMF pointed out in Fig. 1(c). These LMF may reveal more important information for GS, because compared to the large fluctuations, they prove to be more sensitive to relatively small changes in parameters. If one looks closer, there is a slight qualitative difference between the fluctuations in the curves of h_1 and h_3 . The former has a drawn-out shape in the form of a distortion or “bump” that acts to elongate the local cycle and possibly making detection of PS problematical. Thus although the bump shape may keep the relation of GS and its detection, it may hamper the identification of PS by conventional phase frequency methods. The main contribution of this paper is based on the statistics of LMF values such as h_1 and h_3 , which allow the GS transition point to be clearly detected, even when the receiver's time series is only used. The LMF statistics alone make it possible to detect the transition to GS clearly and accurately.

Figure 2 reports the results for the case where $\omega_d < \omega_r$. We give (i) the time series ($\ln z$) of the driver (labeled with $\omega_{d,r}$ values), (ii) the time series of the receiver before GS (labeled with coupling strength), (iii) the time series of the receiver after GS, (iv) the difference between the time series

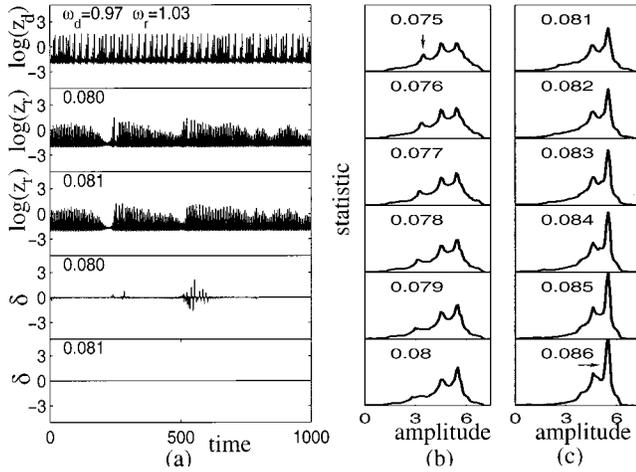


FIG. 2. The result of Rössler with $\omega_d < \omega_r$. (a) From top to bottom: time series [$\log(z)$] of driver, time series of receiver before GS, time series of receiver after GS, difference δ between receiver and auxiliary oscillator before GS, and difference δ after GS; (b) statistical curves (histogram) before GS with coupling strength labeled on the figures (vertical axis is dimensionless, horizontal axis is amplitude of LMF); (c) statistical curve after GS [axis and scale is the same as (b)].

of the receiver and the auxiliary oscillator before GS, and (v) the difference after GS in Fig. 2(a) from top to bottom. One can see that with coupling $g=0.08$, there is no GS between receiver and driver. However, when coupling is increased to $g=0.081$, GS appears, since there is no difference between the signals of the receiver and auxiliary oscillators (we have tested different initial conditions). The same results emerge if GS is detected through calculating the maximal conditional Lyapunov exponent.

In this case, the misfit between driver and receiver oscillators is relatively large, and although there is GS we found it difficult to find indications of PS (i.e., GS does not necessarily imply PS [5]). Conventional phase analysis techniques that require calculating the mean frequency of the receiver and the driver by counting peaks in the time series fail to find PS in this parameter regime. (As we will see, for Rössler oscillators, when the parameter misfit is much lesser than in this example, GS always implies PS and GS is identical with LS.) Now we give the frequency histogram statistics of the amplitudes of the LMF in Fig. 2(b) with the coupling strength lesser than the GS critical point (before GS), and the result in Fig. 2(c) with the coupling strength greater than the GS critical point (after GS). For each coupling strength, we analyzed a time series of length 5×10^6 after having discarded the initial transient of the same length. In Fig. 2(b), with coupling strengths $g=0.075-0.079$, the frequency histograms have three obvious peaks, but those with coupling strengths $g=0.081-0.086$ only have two peaks. There is a clear transition at $g=0.08-0.081$. In the figure with $g=0.075$, the arrow demarks the position of the peak that ultimately disappears. And at the same time, there is a sharp increase of the peak corresponding to the larger-amplitude LMF (see e.g., the arrow in the figure with label 0.086 pointing out the peak). This intriguing change corresponds to the

disappearance of small LMF in the time series of the receiver. In order to make the receiver in GS with the driver, some of the LMFs merge with their adjacent fluctuations.

From Figs. 2(b) and 2(c), we can see this procedure and the difference before and after GS. PS could not be identified for $g < 0.086$, even though GS was often present. At the PS critical point ($g=0.086$), the number of fluctuations within the same length time series of the driver and the receiver should be equal, this corresponds to the conventionally defined PS. When the receiver is in PS with the driver, each peak in the time series of the receiver must have unique corresponding peak in the time series of the driver. This one-to-one relation is not necessary for GS. The small change of the LMF can keep the relation of the GS. Similar result can be found in the region with the parameter mismatch $\omega_r - \omega_d > 0.025$, the typical feature is a sharp decrease of the amount of small LMF and a sharp increase of relative large LMF around the critical point of GS in the amplitude (here absolute value is adopted) statistics of the LMF.

Consider now another example, based on the Lotka-Volterra chaotic foodweb model. The phase synchronization in both Rössler and foodweb models has been studied in Ref. [9]. It was shown that PS has important applications in the study of ecological communities where the spatial coupling of populations can lead to large-scale complex synchronization effects. Here we study the drive-response foodweb oscillators with a moderate parameter misfit where we again find that GS does not necessarily imply PS. The equations are given as follows:

$$\dot{x}_1 = x_1 - 0.2x_1y_1 / (1.0 + 0.05x_1),$$

$$\dot{y}_1 = -b_1y_1 + 0.2x_1y_1 / (1.0 + 0.05x_1) - y_1z_1,$$

$$\dot{z}_1 = -10(z_1 - 0.006) + y_1z_1,$$

$$\dot{x}_2 = x_2 - 0.2x_2y_2 / (1.0 + 0.05x_2),$$

$$\dot{y}_2 = -b_2y_2 + 0.2x_2y_2 / (1.0 + 0.05x_2) + y_2z_2 + g(y_1 - y_2),$$

$$\dot{z}_2 = -10(z_2 - 0.006) + y_2z_2 + g(z_1 - z_2), \quad (2)$$

with parameters $b_1=0.97$ and $b_2=0.9$. The relationship between the numerically determined mean frequency and parameter $b_{1,2}$ has been given in Ref. [9] and is monotonically increasing, making $b_{1,2}$ somewhat analogous to $\omega_{1,2}$ which controls the frequency of the previous Rössler system. As before, since the mean frequency of the driving foodweb oscillator is relatively large compared to the receiving oscillator, LMF [which appears as very small loops in the x - y phase plane Fig. 1(c)] emerges at the point where GS begins to kick in. This contrasts with the situation that the parameter misfit is so small and no LMF appears at the transition to GS. From Fig. 3 (exactly analogous to Fig. 2), one can see that GS occurs when $g=0.119$. Comparing Fig. 3(b), the frequency histograms before GS, and Fig. 3(c), the frequency histograms after GS, the transition to GS is obvious. One can see the height of the LMF peaks very close to zero increases

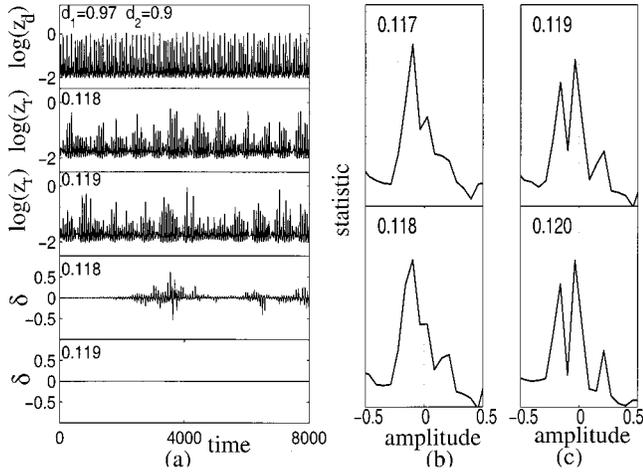


FIG. 3. The result of the second example, foodweb oscillators with $d_1 > d_2$; (a), (b), (c) same as Fig. 2.

abruptly, corresponding to emergence of the extremely small LMF. The GS transition as g is increased from $g = 0.118$ to $g = 0.119$ is seen by comparing the histograms in Fig. 3(b) and 3(c). The result is consistent with that of Fig. 3(a). The bottom two figures give the difference between the receiver and the auxiliary oscillator (coupling strength g is labeled on the figure). When $g = 0.118$ there is no GS, but with $g = 0.119$ GS appears. The transition point may also be confirmed by examination of the maximal conditional Lyapunov exponent, and this result has been tested for various randomly chosen initial conditions.

In Fig. 4, we give another example involving the Rössler oscillators but now with $\omega_d > \omega_r$. One can see that a little larger coupling strength ($g = 0.11$) is required to achieve GS when compared to the case $\omega_d < \omega_r$ ($g = 0.081$), even though the absolute frequency difference $|\omega_d - \omega_r|$ is exactly the same. Thus when the driving oscillator has a smaller frequency ω , one might expect adjacent spikes in the receiving oscillator to somehow merge as it proceeds to synchronize with the driver. If, instead, the faster oscillator drives the slower oscillator, new spikes (loops) should be expected to

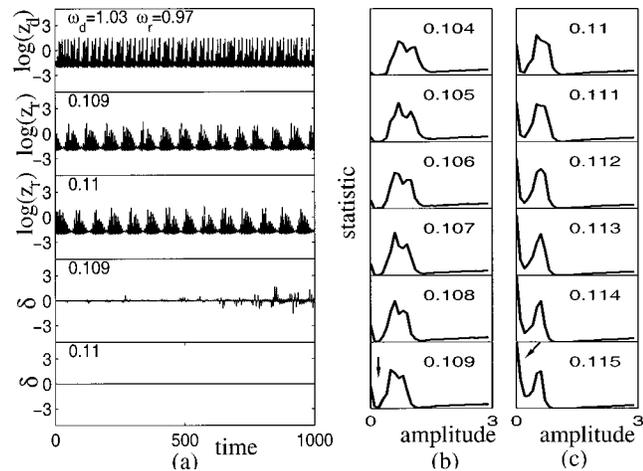


FIG. 4. The result of the third example, Rössler oscillators with $\omega_d > \omega_r$; (a), (b), (c) same as Fig. 2.

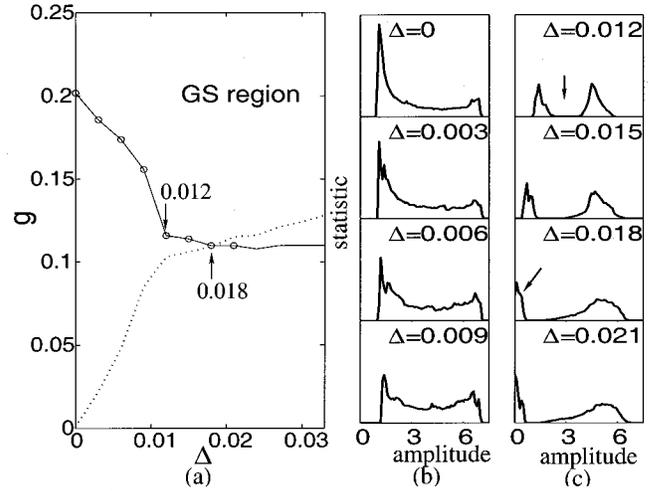


FIG. 5. The statistical change of LMF around the GS critical line ($\omega_d > \omega_r$). (a) The region of the GS between driver and receiver with parameter misfit Δ vs coupling strength g (solid line corresponds to GS critical line with eight circles on it), the statistical curves of LMF at these eight positions are given in (b) and (c) (the dotted line corresponds to the PS critical line), and no PS is observed below this line; (b) statistical curves correspond to $\Delta = 0 \sim 0.009$; (c) statistical curves correspond to $\Delta = 0.012 \sim 0.021$. Two arrows are drawn to point out the dramatic changes.

emerge in the receiver time series. The mechanism is thus very different. For this example (phase-coherent Rössler oscillators), it seems emergence of the adjacent small spikes is easier than emergence of new spikes.

A visual comparison of Figs. 4(b) and 4(c) makes the difference between the GS and non-GS state clear. In Fig. 4(b), the LMF is mainly distributed in the relatively large-amplitude regime. Those LMFs having extremely small amplitudes (almost zero) are either negligible or minor and disconnected from the main part of the histogram. But in Fig. 4(c), where there is GS, the histogram is characterized by a connected histogram with one peak in the middle and one that sits on the vertical axis corresponding to extremely small amplitudes. In Fig. 4(b) with label $g = 0.109$, there is an arrow pointing out the disconnectedness.

A global view of the situation $\omega_d > \omega_r$ in directionally coupled Rössler systems is presented in Fig. 5 (for $\omega_d < \omega_r$ —see also Ref. [5]). Here we outline the characteristics of the LMF statistic in the GS regime but very close to the GS-non-GS bifurcation parameter line [misfit $\Delta = (\omega_d - \omega_r)/2$ vs GS critical coupling strength g]. In Fig. 5(a), two lines are given, where the solid line with circles is the GS critical line and the dotted line corresponds to the PS critical line (counting the peaks in the time series of $\ln z$ to detect mean phase frequency [11]), and no PS is observed below this line. One can see these two lines intersect at $\Delta = 0.018$. Furthermore, note that at $\Delta = 0.012$, there is a qualitative change in direction of the GS bifurcation line. Two interesting changes occur in the histograms of the LMFs. First, when $\Delta \leq 0.012$ the LMF histograms of the GS state are characterized by a continuous and connected distribution. Beyond $\Delta = 0.012$, the histogram suddenly changes

to two disconnected components. Second, when $\Delta > 0.018$, there is a major peak corresponding to the presence of small-amplitude LMFs. Note that this second change occurs exactly where the GS critical line and PS critical line intersect.

In directionally coupled-phase coherent oscillators (eg., Rössler, UPCA [9], Rulkov's circuit [1]) with moderate parameter misfit, when the mean frequency of the driver $\langle \omega_d \rangle$ is obviously larger than that of the receiver $\langle \omega_r \rangle$, intuitively it means that in order to reach GS there must be some mechanism that generates new spikes in the time series of the receiver. We have seen that upon increasing the coupling of the receiver GS to the driver, there is in fact creation of LMFs consisting of small-amplitude spikes in the time series of the receiver. When $\langle \omega_d \rangle$ is smaller than $\langle \omega_r \rangle$, there are mergences of the adjacent small spikes in the time series of the receiver, which is attempting to synchronize to the slower driver. The presence of these LMFs may make it difficult if not impossible to detect PS even it is presented.

In this paper, we analyze GS in directionally coupled phase-coherent oscillators by observing the LMF. Though what we used here is only the distribution of the amplitude of the local-minimal fluctuation, one can also test the distribution of the time between the LMFs. The effectiveness of the present procedure can be intuitively understood as follows. Due to the moderate misfit between the receiver and the driver, the emergence of new topological structures and changes for the "attractor" of the receiver becomes unavoidable. These new structures and changes are preferentially

reflected on the LMFs rather than other large fluctuations [see Figs. 1(b) and 1(c)], because the former have relatively smaller energy ωr^2 if one regards the equations of motion of the receiver as the description of a moving particle (here ω and r are the angle velocity and radius in x - y plane, for definitions, see Ref. [11]). When the particle is moving on the smaller circle with a lower energy, it is easier to secede from its original orbit due to the forcing from the driver oscillator. Furthermore, when a map relationship builds between the driver and the receiver orbit, the critical characteristic of such synchronization is the ability to change the amplitude distribution of LMFs. The method is effective for both $\omega_d > \omega_r$ and $\omega_d < \omega_r$, although one corresponds to emergence of LMF (abrupt increase of extremely small LMF) and the other corresponds to mergence (disappearance) of LMF. By applying this tool, we can find the qualitative changes in the vicinity of the GS critical line, and it is possible to detect GS in practice when the auxiliary scheme fails in situations such as circuit experiments. Moreover, with this method we can distinguish different kinds of GS.

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- [1] N.F. Rulkov, M.M. Sushchik, L.S. Tsimring, and H.D.I. Abarbanel, *Phys. Rev. E* **51**, 980 (1995).
 [2] N.F. Rulkov, V.S. Afraimovich, C.T. Lewis, J.-R. Chazottes, and A. Cordonet, *Phys. Rev. E* **64**, 016217 (2001).
 [3] U. Parlitz, L. Junge, W. Lauterborn, and L. Kocarev, *Phys. Rev. E* **54**, 2115 (1996).
 [4] Z. Liu and S. Chen, *Phys. Rev. E* **56**, 7297 (1997).
 [5] Z. Zheng and G. Hu, *Phys. Rev. E* **62**, 7882 (2000).
 [6] M. Zhan, G.W. Wei, and C.-H. Lai, *Phys. Rev. E* **65**, 036202 (2002).
 [7] T. Wennekers and Frank Pasemann, *Neurocomputing* **38-40**, 1037 (2001).
 [8] B. Blasius, A. Huppert, and L. Stone, *Nature (London)* **399**, 354 (1999).
 [9] B. Blasius and L. Stone, *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **10**, 2361 (2000).
 [10] Z. Zheng, X. Wang, and M. Cross, *Phys. Rev. E* **65**, 056211 (2002).
 [11] A. Pikovsky and M. Rosenblum, *Phys. Rev. E* **64**, 058203 (2001).